

# Spectrum and decays of kaonic hydrogen

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Received: 7 March 2004 /

Published online: 18 May 2004 – © Springer-Verlag / Società Italiana di Fisica 2004

**Abstract.** By using the non-relativistic effective Lagrangian approach to bound states, a complete expression for the isospin-breaking corrections to the energy levels and the decay widths of kaonic hydrogen is obtained up-to-and-including  $O(\alpha, m_d - m_u)$  in QCD. It is demonstrated that, although the leading-order corrections at  $O(\alpha^{1/2}, (m_d - m_u)^{1/2})$  emerging due to the unitarity cusp are huge, they can be expressed solely in terms of the  $KN$  S-wave scattering lengths. Consequently, at leading order, it is possible to derive parameter-free modified Deser-type relations, which can be used to extract the scattering lengths from the hadronic atom data.

## 1 Introduction

The ongoing DEAR experiment [1] at the DAΦNE facility (LNF-INFN) aims at an accurate measurement of the ground-state strong-energy shift and width of kaonic hydrogen and of the strong shift of kaonic deuterium. Preliminary results of the measurements for the kaonic hydrogen have been reported in [2]:

$$\Delta E_1^s = 202 \pm 45 \text{ eV}, \quad \Gamma_1 = 250 \pm 138 \text{ eV}. \quad (1)$$

Here,  $\Delta E_1^s$  stands for the strong-energy-level shift of the ground state of the kaonic hydrogen (total energy shift minus certain electromagnetic contributions), and  $\Gamma_1$  denotes the width of the ground state. It should be pointed out that these results are in contradiction with the earlier measurements [3–6]; see also Fig. 3 below.

The final goal of the DEAR experiment is to extract precise values of the  $KN$  S-wave scattering lengths from the data by using some counterpart of Deser-type relations [7]. Neglecting isospin-breaking corrections altogether, in the case of kaonic hydrogen these relations are given by

$$\Delta E_1^s - \frac{i}{2} \Gamma_1 = -2\alpha^3 \mu_c^2 a_{K^-p}, \quad a_{K^-p} = \frac{1}{2} (a_0 + a_1), \quad (2)$$

where  $\mu_c$  denotes the reduced mass of the  $K^-p$  system, and  $a_0, a_1$  stand for the  $I = 0, 1$  S-wave  $KN$  scattering lengths in QCD in the isospin limit ( $\alpha = 0, m_d = m_u$ ). In addition, our definition of the isospin limit implies that the particle masses in the multiplets are taken to be equal to the charged particle masses in the real world (proton,

$\pi^+, K^+, \dots$ ). Further, in the experimental proposal it has been stated that the precise knowledge of the  $KN$  scattering lengths could allow one to deduce more accurate values for the  $KN$   $\sigma$  terms and the strangeness content of the nucleon. In practice, however, the implementation of the above program might pose a rather big challenge to theory (see, e.g. [8] and references therein). For this reason, in this paper we restrict ourselves to the moderate goal of relating the  $KN$  scattering lengths to the measurable characteristics of kaonic hydrogen at the accuracy that matches the experimental precision. Using these scattering lengths for determining the parameters of the low-energy kaon–nucleon interactions is thus out of the scope of the present paper.

It turns out that the isospin-breaking corrections to the lowest-order relation given by (2) are huge. In particular, these are much larger than their counterparts in pionic hydrogen, or in ponium (typically, a few percent). This can be immediately seen, e.g., from Table 1 (see below) by comparing the entries in the same column. The reason for this qualitative difference will be discussed. In addition, the existing predictions in the literature, most of which are done in the framework of potential scattering theory (see [9–12] for an incomplete list of the earlier work on the subject), are anything but consistent with each other. In particular, although large corrections have been predicted in some of these papers, these effects have not been treated systematically – e.g., it is not always clear whether all possible large corrections are taken into account. Needless to say, all this could make the interpretation of the results of the accurate measurements of the DEAR experiment a difficult task.

The aim of the present paper is to obtain the formal relation between the energy shift and the width of kaonic hy-

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drogen, and the  $KN$  scattering lengths up-to-and-including isospin-breaking effects at  $O(\alpha, m_d - m_u)^1$  in QCD, by using the systematic approach to the bound-state problem based on the non-relativistic effective Lagrangians. In the past, this method has already been applied to study pionium and kaonium [13–15], as well as pionic hydrogen [16–18]. It happens to be a very useful and convenient approach to describe the spectrum and decays of this sort of bound states. We shall see that the approach is universal: the treatment of kaonic hydrogen closely follows the pattern of pionic hydrogen.

## 2 Formalism

As it was already pointed out, the quantities that we are aiming to extract from the data on kaonic hydrogen, are the S-wave  $KN$  scattering lengths  $a_0, a_1$  evaluated in QCD in the isospin limit, i.e. in the absence of the electromagnetic interactions and at  $m_d = m_u$ . Note that we avoid using the threshold scattering amplitude calculated with physical hadron masses but in the absence of electromagnetic effects, which is sometimes encountered in the literature (see e.g., [19, 20]). The reason for this is that this quantity cannot be consistently made ultraviolet-finite to all orders in chiral perturbation theory (ChPT)<sup>2</sup>. Further, the isospin-breaking effects are parameterized in terms of  $\alpha$  and  $m_d - m_u$ . It is convenient to introduce a correlated counting of these effects, defining a formal parameter  $\delta \sim \alpha \sim m_d - m_u$  [16]. Equation (2) is then valid up-to-and-including  $O(\delta^3)$  in isospin breaking, and to all orders in the chiral expansion for the quantity  $a_0 + a_1$  which is present in this expression. In the present paper we modify the relation (2), including all terms of order  $\delta^{7/2}$  and  $\delta^4$ .

In order to construct a non-relativistic Lagrangian that can describe the spectrum of kaonic hydrogen at  $O(\delta^4)$ , we note the following.

(i) The only states that are degenerate in mass with the  $K^-p$  state in the isospin limit  $\delta \rightarrow 0$ , are the states  $K^-p + n\gamma$ ,  $\bar{K}^0n + n'\gamma$ , with  $n, n' = 0, 1, \dots$ . We explicitly “resolve” only these states in our non-relativistic theory, whereas the effect of other intermediate states, whose mass is not degenerate with that of the  $K^-p$  state in the isospin limit, is included in the couplings of the non-relativistic effective Lagrangian.<sup>3</sup>

<sup>1</sup> We use throughout the Landau symbols  $O(x)$  [ $o(x)$ ] for quantities that vanish like  $x$  [faster than  $x$ ] when  $x$  tends to zero. Furthermore, it is understood that this holds modulo logarithmic terms, i.e. we write also  $O(x)$  for  $x \ln x$ .

<sup>2</sup> Consider the loop contributions to the scattering amplitude with both charged and neutral particles running inside the diagrams. The divergent parts which are generated by these loops depend on charged and the neutral particle masses. Since the mass difference contains the electromagnetic piece proportional to  $e^2$ , in order to cancel all divergences one needs, along with the “strong” counterterm Lagrangian, the “electromagnetic” counterterms as well. The latter was, however, ruled out from the beginning.

<sup>3</sup> For the treatment of the  $n\gamma$  intermediate state in pionic hydrogen, which is similar in spirit to the approach used here, see [18].

In particular, the  $SU(3)$  breaking scale  $m_s - m_u$  counts at  $O(\delta^0)$  in our approach. As a result, all effects which are non-analytic in the parameter  $\delta$  – e.g. containing  $\sqrt{\delta}$  or  $\ln \delta$ , should be produced by the loop expansion in the non-relativistic theory. To the contrary, the generic couplings  $g_i$  of the non-relativistic Lagrangian are regular functions of  $\delta$  and can be expanded in Taylor series:

$$g_i = g_i^{(0)} + \alpha g_i^{(1)} + (m_d - m_u) g_i^{(2)} + O(\delta^2). \quad (3)$$

(ii) The couplings  $\tilde{d}_i$  that describe the  $KN$  scattering in the tree approximation (see (4)) are complex. The imaginary parts of the  $\tilde{d}_i$  can be related through the unitarity condition to the transition cross sections of the  $KN$  initial state into the different inelastic channels. In this case, there exist open strong channels – e.g.  $\pi\Sigma, \pi^0\Lambda$ , etc. The mass gap between these shielded two-particle states and the  $KN$  state is determined by the  $SU(3)$  breaking scale. Consequently, the couplings  $\tilde{d}_i$  are complex already at  $O(\delta^0)$ . This is different from the case of the pionic hydrogen, where the imaginary part of the effective two-pion–two-nucleon couplings is of order  $\delta$ .

(iii) The leading strong decay channel in the case of pionic hydrogen is  $\pi^0n$ . The phase space for this decay channel is proportional to  $(m_p + M_{\pi^+} - m_n - M_{\pi^0})^{1/2}$  and is thus suppressed by a factor  $\delta^{1/2}$ . For this reason, the ratio of the decay widths into the leading electromagnetic channel  $n\gamma$ , and into the  $\pi^0n$  channel counts as order  $\delta^{1/2}$  only. Numerically, the branching ratio into the  $n\gamma$  channel amounts up to  $\sim 40\%$  in the total decay width. In contrast to this, in the case of kaonic hydrogen this branching ratio counts as  $O(\delta)$ . The measured branching ratio into the leading  $A\gamma, \Sigma\gamma$  channels is much less than 1% [21] (the theoretical description of this quantity by using chiral Lagrangians [12, 22] gives a result consistent with the experiment by order of magnitude). Consequently, the perturbative treatment of the effects due to these channels, as it is carried out in this paper, is justified.<sup>4</sup>

(iv) Within our approach, it is sufficient to deal with the sub-threshold  $\Lambda(1405)$  resonance indirectly, through the (large)  $KN$  scattering lengths. The reason for this simplification is that the mass gap counts at  $O(\delta^0)$  in our counting of the isospin-breaking effects, i.e. the effect occurs at a “hard scale” and should be included in the effective couplings.

(v) The  $\bar{K}^0n$  system is a bit heavier than  $K^-p$ . This simple fact has dramatic consequences on the size of isospin-breaking corrections, which is nothing but the well-known cusp effect (note that the cusp effect is also the dominant isospin-breaking effect in some other low-energy processes, e.g. in neutral pion photo-production off nucleons [23]). Namely, the loop with the  $\bar{K}^0n$  intermediate state at threshold in the non-relativistic theory is proportional to  $(m_n + M_{\bar{K}^0} - m_p - M_{K^+})^{1/2} \sim \sqrt{\delta}$  and is *real*. Its counterpart for the pionic hydrogen case is *purely imaginary*. This means that, in the case of pionic hydrogen, the corrections to the *real* quantities – the energy shift

<sup>4</sup> We thank E. Oset for interesting discussions on this issue.

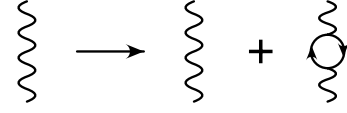
and width – cannot contain a contribution from a single neutral loop. Only the product of two loops, which is a real quantity, can contribute – therefore, the corrections start at order  $\delta$  with respect to the leading-order term. To the contrary, the corrections for kaonic hydrogen can contain a single neutral loop. Due to this, the isospin-breaking corrections to the Deser formula for kaonic hydrogen start at  $O(\sqrt{\delta})$  and are much larger than their counterparts for pionic hydrogen.

Despite the differences between pionic and kaonic hydrogen that were discussed above, one may apply exactly the same formal approach in both cases to the calculation of the bound state spectra. Below, we closely follow the path outlined in [16]. The effective non-relativistic Lagrangian is given by

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \psi^\dagger \left\{ iD_t - m_p + \frac{\mathcal{D}^2}{2m_p} + \frac{\mathcal{D}^4}{8m_p^3} + \dots \right. \\
 & - c_p^F \frac{e\sigma\mathbf{B}}{2m_p} - c_p^D \frac{e(\mathbf{DE} - \mathbf{ED})}{8m_p^2} \\
 & \left. - c_p^S \frac{ie\sigma(\mathcal{D} \times \mathbf{E} - \mathbf{E} \times \mathcal{D})}{8m_p^2} + \dots \right\} \psi \\
 & + \chi^\dagger \left\{ i\partial_t - m_n + \frac{\nabla^2}{2m_n} + \frac{\nabla^4}{8m_n^3} + \dots \right\} \chi \\
 & + \sum_{\pm} (K^\pm)^\dagger \left\{ iD_t - M_{K^\pm} + \frac{\mathbf{D}^2}{2M_{K^\pm}} + \frac{\mathbf{D}^4}{8M_{K^\pm}^3} + \dots \right. \\
 & \left. \mp c_K^R \frac{e(\mathbf{DE} - \mathbf{ED})}{6M_{K^\pm}^2} + \dots \right\} K^\pm \\
 & + (\bar{K}^0)^\dagger \left\{ i\partial_t - M_{\bar{K}^0} + \frac{\nabla^2}{2M_{\bar{K}^0}} + \frac{\nabla^4}{8M_{\bar{K}^0}^3} + \dots \right\} \bar{K}^0 \\
 & + \tilde{d}_1 \psi^\dagger \psi (K^-)^\dagger K^- + \tilde{d}_2 (\psi^\dagger \chi (K^-)^\dagger \bar{K}^0 + \text{h.c.}) \\
 & + \tilde{d}_3 \chi^\dagger \chi (\bar{K}^0)^\dagger \bar{K}^0 + \dots \quad (4)
 \end{aligned}$$

Here,  $F_{\mu\nu}$  stands for the electromagnetic field strength tensor (we work in the Coulomb gauge). Further,  $\psi$ ,  $\chi$ ,  $K^\pm$  and  $\bar{K}^0$  denote the non-relativistic field operators for the proton, neutron, charged and neutral kaon fields, and  $\mathcal{D}_t\psi = \partial_t\psi - ieA_0\psi$ ,  $\mathcal{D}\psi = \nabla\psi + ie\mathbf{A}\psi$ ,  $D_t K^\pm = \partial_t K^\pm \mp ieA_0 K^\pm$ ,  $\mathbf{D}K^\pm = \nabla K^\pm \pm ie\mathbf{A}K^\pm$  are the covariant derivatives acting on the proton and charged pion fields, respectively. The ellipsis stands for the higher-dimensional operators and the UV counterterms. The values of the couplings  $c_i$  can be read off from the matching condition for the kaon and nucleon electromagnetic form-factors:  $c_p^F = 1 + \mu_p$ ,  $c_p^D = 1 + 2\mu_p + \frac{4}{3} m_p^2 \langle r_p^2 \rangle$ ,  $c_p^S = 1 + 2\mu_p$ ,  $c_K^R = M_{K^\pm}^2 \langle r_K^2 \rangle$ , where  $\mu_p$  denotes the anomalous magnetic moment of the proton, and  $\langle r_p^2 \rangle$ ,  $\langle r_K^2 \rangle$  stand for the squared charge radii of the proton and the charged kaon, respectively.

Formally, the vacuum polarization contributions to the energy shift and width start at  $O(\alpha^5)$ . It is, however, well known that the electronic vacuum polarization contribution is amplified by powers of the factor  $\mu_c/m_e \sim 10^3$ , with  $m_e$  being the electron mass. It is convenient to count the quantity  $\eta \doteq \alpha\mu_c/m_e$  as  $O(\delta^0)$ . Then, the leading-



**Fig. 1.** Modification of the timelike component of the free photon propagator due to the electron vacuum polarization

order contribution starts already at  $O(\delta^3)$ , and the next-to-leading order contribution (interference of strong interactions with the vacuum polarization) comes at  $O(\delta^4)$ . In this paper, we explicitly include these contributions in the expression of the energy. Note that at the accuracy considered here, the whole effect of the vacuum polarization reduces to the modification of the timelike component of the free photon propagator by an electron loop (see Fig. 1):

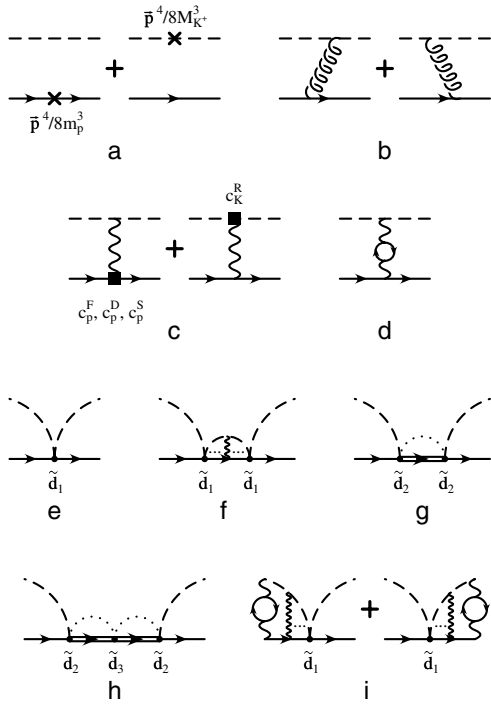
$$\begin{aligned}
 D_{00}(k) = & -\frac{1}{k^2} \quad (5) \\
 \rightarrow & -\frac{1}{k^2} - \frac{\alpha}{3\pi} \int_{4m_e^2}^{\infty} \frac{ds}{s+k^2} \frac{1}{s} \left(1 + \frac{2m_e^2}{s}\right) \sqrt{1 - \frac{4m_e^2}{s}}.
 \end{aligned}$$

### 3 Energy shift and width of kaonic hydrogen

The energy spectrum of kaonic hydrogen is obtained by using Rayleigh–Schrödinger perturbation theory (for details, see [16]; see also [24] for the derivation of the energy shift for a generic excited state of the  $\pi K$  atom by using non-relativistic Lagrangians). Namely, at the first step one constructs the full Hamiltonian  $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_C + \mathbf{V}$  from the Lagrangian given by (4). In this expression,  $\mathbf{H}_0$  stands for the free non-relativistic Hamiltonian of the  $KN$  pair,  $\mathbf{H}_C$  denotes the pure Coulomb interaction between  $K^-$  and the proton, and the rest of the interaction is included in the operator  $\mathbf{V}$  which is treated as a perturbation. The general solution of the unperturbed Schrödinger equation with the Hamiltonian  $\mathbf{H}_0 + \mathbf{H}_C$ , that corresponds to the quantum-mechanical Coulomb problem for the bound system of spin-0 and spin-1/2 particles, is characterized by the quantum numbers  $n = 1, 2, \dots$ ,  $j = \frac{1}{2}, \frac{3}{2}, \dots$ ,  $m = -j, \dots, j$  and  $l = j \pm \frac{1}{2}$ ,

$$\begin{aligned}
 (\mathbf{H}_0 + \mathbf{H}_C)|\Psi_{nljm}(\mathbf{P})\rangle & = \bar{E}_n(\mathbf{P})|\Psi_{nljm}(\mathbf{P})\rangle, \\
 \bar{E}_n(\mathbf{P}) & = m_p + M_{K^\pm} + \frac{\mathbf{P}^2}{2(m_p + M_{K^\pm})} - \frac{\mu_c \alpha^2}{2n^2} \\
 & \doteq \bar{E}_n + \frac{\mathbf{P}^2}{2(m_p + M_{K^\pm})}, \\
 |\Psi_{nljm}(\mathbf{P})\rangle & = \sum_s \int \frac{d^3\mathbf{q}}{(2\pi)^3} \langle jm|l(m-s)\frac{1}{2}s\rangle \\
 & \quad \times Y_{l(m-s)}(\mathbf{q}) \Psi_{nl}(|\mathbf{q}|) |\mathbf{P}, \mathbf{q}, s\rangle, \\
 |\mathbf{P}, \mathbf{q}, s\rangle & = b^\dagger(\mu_1 \mathbf{P} + \mathbf{q}, s) a^\dagger(\mu_2 \mathbf{P} - \mathbf{q}) |0\rangle. \quad (6)
 \end{aligned}$$

Here,  $\mu_1 = m_p/(m_p + M_{K^\pm})$ ,  $\mu_2 = M_{K^\pm}/(m_p + M_{K^\pm})$ ,  $a^\dagger$ ,  $b^\dagger$  stand for the creation operators for the non-relativistic



**Fig. 2.** The set of diagrams contributing to the energy shift of the kaonic hydrogen up-to-and-including  $O(\delta^4)$ . Solid, dashed, double, dotted, wiggly and spring lines correspond to the proton,  $K^-$ , neutron,  $\bar{K}^0$ , Coulomb and transverse photons, respectively. The electrons run in the closed loops shown in diagrams **d** and **i**. The diagrams **f** and **i** contain Coulomb ladders – the contributions with 0, 1, 2, ... Coulomb photons exchanged

proton and  $K^-$ ,  $Y_{lm}$  and  $\langle jm|l(m-s)\frac{1}{2}s\rangle$  are the spherical harmonics and the pertinent Clebsch–Gordan coefficients, respectively, and  $\Psi_{nl}(|\mathbf{q}|)$  denotes the radial Coulomb wave function, which depends on the magnitude of the relative momentum  $|\mathbf{q}|$ . We further define the “pole-removed” Coulomb Green function  $\hat{\mathbf{G}}_{nlj}(z)$  and the elastic transition operator  $\mathbf{M}_{nlj}(z)$ :

$$\begin{aligned} \mathbf{G}_0(z) &= \frac{1}{z - \mathbf{H}_0}, & \mathbf{G}(z) &= \frac{1}{z - \mathbf{H}_0 - \mathbf{H}_C}, \\ \hat{\mathbf{G}}_{nlj}(z) &= \mathbf{G}(z) - \sum_m \int \frac{d^3\mathbf{P}}{(2\pi)^3} \frac{|\Psi_{nljm}(\mathbf{P})\rangle\langle\Psi_{nljm}(\mathbf{P})|}{z - \bar{E}_n(\mathbf{P})}, \\ \mathbf{M}_{nlj}(z) &= \mathbf{V} + \mathbf{V}\hat{\mathbf{G}}_{nlj}(z)\mathbf{M}_{nlj}(z). \end{aligned} \quad (7)$$

By using Feshbach’s formalism [25], one can show that the (complex) energy shift of the level characterized by the quantum numbers  $njl$  is given by

$$\begin{aligned} \Delta E_{nlj} &= \langle\Psi_{nljm}|\mathbf{m}_{nlj}(\bar{E}_n)|\Psi_{nljm}\rangle + o(\delta^4), \\ &\langle\Psi_{nljm}(\mathbf{P})|\mathbf{M}_{nlj}(z)|\Psi_{nljm}(\mathbf{0})\rangle \\ &\doteq (2\pi)^3\delta^3(\mathbf{P})\langle\Psi_{nljm}|\mathbf{m}_{nlj}(z)|\Psi_{nljm}\rangle. \end{aligned} \quad (8)$$

In order to evaluate the energy shift up-to-and-including  $O(\delta^4)$ , it suffices to include only the diagrams (a)–(i) shown in Fig. 2 in the calculation of the operator  $\mathbf{M}_{nlj}(z)$ . The resulting expression is then sandwiched between Coulomb

wave functions. We stress that none of the other diagrams that can be constructed from the Lagrangian (4), nor any possible contribution from higher-dimensional operators that are not explicitly displayed in (4), contributes to the energy shift at order  $\delta^4$ . Here we do not provide the details of the calculations; only the final result is given. It is convenient to introduce a well-defined splitting of the energy shift into the “electromagnetic” and “strong” parts,<sup>5</sup> where the former does not contain the couplings  $\tilde{d}_i$ :

$$\begin{aligned} \Delta E_{nlj} &= \Delta E_{nlj}^{em} + \delta_{l0} \left( \Delta E_n^s - \frac{i}{2} \Gamma_n \right) + o(\delta^4), \\ \Delta E_{nlj}^{em} &= -\frac{m_p^3 + M_{K^+}^3}{8m_p^3 M_{K^+}^3} \left( \frac{\alpha\mu_c}{n} \right)^4 \left\{ \frac{4n}{l + \frac{1}{2}} - 3 \right\} \\ &\quad - \frac{\alpha^4 \mu_c^3}{4m_p M_{K^+} n^4} \left\{ -4n\delta_{l0} - 4 + \frac{6n}{l + \frac{1}{2}} \right\} \\ &\quad + \frac{2\alpha^4 \mu_c^3}{n^4} \left( \frac{c_p^F}{m_p M_{K^+}} + \frac{c_p^S}{2m_p^2} \right) \\ &\quad \times \left\{ \frac{n}{2l+1} - \frac{n}{2j+1} - \frac{n}{2} \delta_{l0} \right\} \\ &\quad + \frac{4\alpha^4 \mu_c^3}{n^3} \delta_{l0} \left( \frac{c_p^D}{8m_p^2} + \frac{c_K^R}{6M_{K^+}^2} \right) + \Delta E_{nl}^{(d)}, \\ \Delta E_n^s - \frac{i}{2} \Gamma_n &= -\frac{\alpha^3 \mu_c^3}{\pi n^3} \left\{ \tilde{d}_1 - \frac{\alpha\mu_c^2}{2\pi} \tilde{d}_1^2 (\chi + s_n(\alpha)) \right. \\ &\quad \left. - \tilde{d}_2^2 \frac{\mu_0 q_0}{2\pi} + \tilde{d}_2^2 \tilde{d}_3 \left( \frac{\mu_0 q_0}{2\pi} \right)^2 \right\} + \Delta E_n^{(i)}, \\ \chi &= \mu^{2(d-4)} \left( \frac{1}{d-4} - \Gamma'(1) - \ln 4\pi \right) \\ &\quad + \ln \frac{(2\mu_c)^2}{\mu^2} - 1, \\ s_n(\alpha) &= 2 \left( \psi(n) - \psi(1) - \frac{1}{n} + \ln \alpha - \ln n \right), \end{aligned} \quad (9)$$

and  $\mu_0 = m_n M_{\bar{K}^0} / (m_n + M_{\bar{K}^0})$ ,  $q_0 = (2\mu_0(m_n + M_{\bar{K}^0} - m_p - M_{K^+}))^{1/2}$ ,  $\psi(x) \doteq \Gamma'(x)/\Gamma(x)$ . Throughout this paper, we use dimensional regularization to tame both UV and IR divergences. In the above formulae,  $d$  stands for the number of space-time dimensions and  $\mu$  denotes the scale of the dimensional regularization. As a check on our calculations, we have verified that the electromagnetic contributions in the above formulae that come from the diagrams (a) + (b) + (c) reproduce the known result of [26] in the limit  $\langle r_p^2 \rangle = \langle r_K^2 \rangle = 0$ . Moreover, with  $c_p^F = c_p^S = 0$  the above expressions reproduce the energy shift in the bound state of two spin-0 particles [24].

<sup>5</sup> This naming scheme should not be understood literally. For example, the “electromagnetic” contribution depends on the electromagnetic radii of the proton and  $K^-$ , which are determined mainly by strong interactions. On the other hand, there are electromagnetic corrections to the couplings  $\tilde{d}_i$ .

The contributions due to the vacuum polarization  $\Delta E_{nl}^{(d)}$ ,  $\Delta E_n^{(i)}$  have been evaluated e.g. in [27], in the same approach as used in the present paper (see also [13, 28] for the related discussion). At the order of accuracy we are working, these contributions do not depend on the total angular momentum  $j$ . We do not display here the (quite voluminous) general result for any  $n$  and  $l$ . The expression for  $\Delta E_{nl}^{(d)}$  is given in (3) of [27]. Furthermore, one may write  $\Delta E_n^{(i)} = -(\alpha^3 \mu_c^3 / \pi n^3) \tilde{d}_1 \delta_n^{\text{vac}} + o(\delta^4)$ , and relate this quantity to the correction to the bound-state wave function at the origin due to the vacuum polarization effect  $\delta_n^{\text{vac}} = 2 \delta \Psi_n(0) / \Psi_n(0)$ . In [27], this correction has been explicitly evaluated for the ground state (see (6) and Table II of this paper), although the method used in this paper enables one to make calculations for any excited level. Here, it is important to stress that the “electromagnetic” contributions from diagrams (a) + (b) + (c) + (d), which have to be unambiguously identified and systematically evaluated up-to-and-including  $O(\delta^4)$ , are only used for determining the so-called “strong shift” (see (2)) from which the information about the  $KN$  scattering lengths is extracted. Namely, the strong shift is defined as a difference between the total energy shift and the electromagnetic shift. In the rest of the paper we deal with the strong shift only.

Equations (9) do not solve our problem completely: the energy shift is expressed in terms of the effective couplings  $\tilde{d}_i$  which have still to be related to the observable quantities. As in [15, 16, 18, 24], this goal is achieved by performing the matching for the  $KN$  scattering amplitudes in the vicinity of threshold. In the absence of isospin breaking one immediately gets

$$\tilde{d}_1^{(0)} = \tilde{d}_3^{(0)} = \frac{\pi}{\mu_c} (a_0 + a_1), \quad \tilde{d}_2^{(0)} = \frac{\pi}{\mu_c} (a_1 - a_0), \quad (10)$$

where  $\tilde{d}_i = \tilde{d}_i^{(0)} + O(\delta)$ . However, as one sees from (9), in order to evaluate  $\Delta E_{nlj}$  at  $O(\delta^4)$ ,  $\tilde{d}_1$  should be known at  $O(\delta)$  (for  $\tilde{d}_2$ ,  $\tilde{d}_3$  the accuracy of (10) suffices). At the required precision, the quantity  $\tilde{d}_1$  can be determined from matching to the  $K^-p$  threshold elastic scattering amplitude in the presence of electromagnetic and strong isospin-breaking effects at  $O(\delta)$  – the corresponding procedure is described in detail in [16, 17]. At the first step, one removes the one-photon exchange from the spin-non-flip part of the relativistic scattering amplitude for the process  $p(p) + K^-(q) \rightarrow p(p') + K^-(q')$ :

$$T_{KN} = \bar{u}(p') \left\{ \tilde{D}(s, t) - \frac{1}{4m_p} [\not{q}', \not{q}] \tilde{B}(s, t) \right\} u(p),$$

$$\tilde{D}'(s, t) = \tilde{D}(s, t) - e^2 F_K(t) F_1(t) (s - u) / (2m_p t), \quad (11)$$

where  $F_K(t)$ ,  $F_1(t)$  denote the kaon electromagnetic and the nucleon Dirac formfactors, respectively, and  $s, t, u$  are the usual Mandelstam variables. The quantity  $\tilde{D}'(s, t)$  is singular at threshold, as the magnitude of the relative three-momentum of the proton and kaon in the CM frame  $|\mathbf{p}|$  vanishes. At  $O(\delta)$ , the structure of this singularity is given

by [16, 17]

$$\begin{aligned} & e^{-2ia\theta_C(|\mathbf{p}|)} \tilde{D}'(s, t) \Big|_{|\mathbf{p}| \rightarrow 0} \\ &= \frac{\tilde{B}_1}{|\mathbf{p}|} + \tilde{B}_2 \ln \frac{|\mathbf{p}|}{\mu_c} + \mathcal{T}_{KN} + O(|\mathbf{p}|), \end{aligned} \quad (12)$$

where  $\theta_C(|\mathbf{p}|)$  denotes the (dimensionally regularized infrared-divergent) Coulomb phase

$$\begin{aligned} & \theta_C(|\mathbf{p}|) \\ &= \frac{\mu_c}{|\mathbf{p}|} \mu^{d-4} \left( \frac{1}{d-4} - \frac{1}{2} (\Gamma'(1) + \ln 4\pi) + \ln \frac{2|\mathbf{p}|}{\mu} \right). \end{aligned} \quad (13)$$

In this normalization, the S-wave  $KN$  scattering lengths and the threshold amplitude  $\mathcal{T}_{KN}$  in the isospin limit are related by

$$\mathcal{T}_{KN} = 4\pi \left( 1 + \frac{M_{K^+}}{m_p} \right) \frac{1}{2} (a_0 + a_1) + O(\sqrt{\delta}). \quad (14)$$

The quantity  $\mathcal{T}_{KN}$  should be matched to its non-relativistic counterpart  $\mathcal{T}_{KN}^{\text{NR}}$ , written in terms of the couplings  $\tilde{d}_i$ . A direct calculation with the Lagrangian (4), which is carried out in a similar way as in [16], yields

$$\mathcal{T}_{KN}^{\text{NR}} = \tilde{d}_1 - \tilde{d}_2^2 \frac{\mu_0 q_0}{2\pi} + \tilde{d}_2^2 \tilde{d}_3 \left( \frac{\mu_0 q_0}{2\pi} \right)^2 - \tilde{d}_1^2 \frac{\alpha \mu_c^2}{2\pi} (\chi - 2\pi i). \quad (15)$$

The matching condition  $2M_{K^+} \mathcal{T}_{KN}^{\text{NR}} = \mathcal{T}_{KN}$  enables one to determine the coupling  $\tilde{d}_1$  at the required accuracy. Substituting this value of  $\tilde{d}_1$  into the expression for the strong shift, we finally get the formula in terms of the observable quantities, which contains all isospin-breaking terms up-to-and-including  $O(\delta^4)$ :

$$\begin{aligned} \Delta E_n^{\text{S}} - \frac{i}{2} \Gamma_n &= -\frac{\alpha^3 \mu_c^3}{2\pi M_{K^+} n^3} \\ &\times \mathcal{T}_{KN} \left\{ 1 - \frac{\alpha \mu_c^2}{4\pi M_{K^+}} \mathcal{T}_{KN} (s_n(\alpha) + 2\pi i) + \delta_n^{\text{vac}} \right\}. \end{aligned} \quad (16)$$

Although (16) formally solves the problem stated in the introduction, it is still not well suited for the analysis of the experimental data. The reason for this is clear from (15). There it is immediately seen that the unitarity correction from the  $\bar{K}^0 n$  bubble (second term in the RHS of this equation), whose counterpart in the bound-state sector is depicted in Fig. 2g, starts to contribute to the isospin-breaking part of  $\mathcal{T}_{KN}^{\text{NR}}$  and  $\mathcal{T}_{KN}$  at  $O(\sqrt{\delta})$  (the quantity  $q_0$  is of order  $\sqrt{\delta}$ ). The situation here differs from the pionic hydrogen case where the counterpart of the quantity  $q_0$  is imaginary because  $m_p + M_{\pi^+} > m_n + M_{\pi^0}$ , and the imaginary part of  $\mathcal{T}_{\pi N}$  starts at  $O(\sqrt{\delta})$ , not  $O(1)$ . Consequently, in the  $\pi N$  case the analog of (14) for the real part of the scattering amplitude contains corrections at order  $\delta$  and not  $\sqrt{\delta}$ . Exactly the above-described effect, which is nothing but the unitarity cusp in the  $K^-p$  elastic amplitude, is the source of the huge isospin-breaking corrections in the

energy shift of kaonic hydrogen, which were mentioned in the introduction.

The above problem can be solved in the following manner. From (15) we see that only the unitarity correction to the quantity  $\mathcal{T}_{KN}^{\text{NR}}$  behaves like  $\sqrt{\delta}$ , and all other corrections, including corrections in the couplings  $\tilde{d}_i$ , start at  $O(\delta)$  and are regular in  $\delta$  at this order. On the other hand, from (10) it is seen that the quantity  $\tilde{d}_2$  can be written in terms of the scattering lengths, up to the terms of order  $\delta$ . From this one concludes that the corrections at  $O(\sqrt{\delta})$ , albeit big, are expressed only in terms of the same scattering lengths as are already present in the Deser-type relations at the leading order,<sup>6</sup> and the structure-dependent corrections start at  $O(\delta)$ . This counting can be implemented in the Deser formula, e.g. as follows. We sum up any number of strong neutral bubbles shown in Fig. 2g, since the first term in this expansion contains exactly the desired singular piece with  $\sqrt{\delta}$ . Further, instead of (14), we write

$$\begin{aligned}\mathcal{T}_{KN} &= \mathcal{T}_{KN}^{(0)} + \frac{i\alpha\mu_c^2}{2M_{K^+}} (\mathcal{T}_{KN}^{(0)})^2 + \delta\mathcal{T}_{KN} + o(\delta), \\ \mathcal{T}_{KN}^{(0)} &= 4\pi \left(1 + \frac{M_{K^+}}{m_p}\right) \frac{\frac{1}{2}(a_0 + a_1) + q_0 a_0 a_1}{1 + \frac{q_0}{2}(a_0 + a_1)}.\end{aligned}\quad (17)$$

The above equation is nothing but the definition of  $\delta\mathcal{T}_{KN}$ , and our statement amounts to

$$\delta\mathcal{T}_{KN} = \alpha t_1 + (m_d - m_u) t_2 + o(\delta), \quad (18)$$

where  $t_1, t_2$  are functions of  $\hat{m} = \frac{1}{2}(m_u + m_d)$ ,  $m_s$  and  $\Lambda_{\text{QCD}}$ . For the actual calculation of  $\delta\mathcal{T}_{KN}$  one has, e.g. as in the  $\pi N$  case, to use the information about the underlying dynamics which is contained in the low-energy effective chiral meson–baryon Lagrangians. With the definition given in (17), the formula for the strong shift looks similar to that for the case of the pionic hydrogen

$$\begin{aligned}\Delta E_n^s - \frac{i}{2}\Gamma_n &= -\frac{\alpha^3\mu_c^3}{2\pi M_{K^+} n^3} \\ &\times (\mathcal{T}_{KN}^{(0)} + \delta\mathcal{T}_{KN}) \left\{1 - \frac{\alpha\mu_c^2 s_n(\alpha)}{4\pi M_{K^+}} \mathcal{T}_{KN}^{(0)} + \delta_n^{\text{vac}}\right\},\end{aligned}\quad (19)$$

where  $\mathcal{T}_{KN}^{(0)}$ ,  $\delta\mathcal{T}_{KN}$  are given by (17), and  $\delta\mathcal{T}_{KN} = O(\delta)$ . This is the final formula, which is best suited for the analysis of the experimental data.

## 4 Results, discussion and higher order corrections

The isospin-breaking corrections at  $O(\sqrt{\delta})$  that are contained in the relation of  $\mathcal{T}_{KN}^{(0)}$  to the S-wave  $KN$  scattering lengths  $a_0, a_1$  are numerically by far the dominant ones. These corrections have been derived more than 40 years

<sup>6</sup> The same is true for the non-analytic corrections from (16), which are proportional to  $\alpha \ln \alpha$ .

**Table 1.** Expansion of the  $KN$  scattering amplitude  $\mathcal{T}_{KN}^{(0)}$  in powers of  $q_0$ . Scattering lengths and amplitudes are given in fm, and  $\mathcal{N} \doteq (4\pi(1 + M_{K^+}/m_p))^{-1}$

	[20]	[30]
	$a_0 = -1.31 + 1.24i$	$a_0 = -1.70 + 0.68i$
	$a_1 = 0.26 + 0.66i$	$a_1 = 0.37 + 0.60i$
$\mathcal{N}\mathcal{T}_0^{(0)}$	$-0.52 + 0.95i$	$-0.66 + 0.64i$
$\mathcal{N}\mathcal{T}_1^{(0)}$	$-0.68 + 1.09i$	$-0.98 + 0.66i$
$\mathcal{N}\mathcal{T}_2^{(0)}$	$-0.67 + 1.15i$	$-1.04 + 0.73i$
$\mathcal{N}\mathcal{T}_3^{(0)}$	$-0.65 + 1.16i$	$-1.04 + 0.75i$
$\mathcal{N}\mathcal{T}_\infty^{(0)}$	$-0.65 + 1.15i$	$-1.03 + 0.76i$

ago [9, 10] by using the  $K$ -matrix formalism. However, this piece of information should be still supplemented by the arguments in favor of the conjecture that the remaining corrections are small, i.e.  $\delta\mathcal{T}_{KN} = O(\delta)$ , as done in the present paper.

In order to get a feeling for how big the corrections to the Deser formula can be, we have done a simple exercise. In Table 1 we list the results of the expansion of the quantity  $\mathcal{T}_{KN}^{(0)}$  in powers of  $q_0 \sim \sqrt{\delta}$ , so that  $\mathcal{T}_m^{(0)}$  denotes  $\mathcal{T}_{KN}^{(0)}$  calculated up-to-and-including  $O(q_0^m)$  and  $\mathcal{T}_{KN}^{(0)} \doteq \mathcal{T}_\infty^{(0)}$ . The values for the scattering lengths that are needed in these calculations are taken from [20].<sup>7</sup> In these papers, the  $KN$  scattering amplitudes are obtained by iterating tree-level diagrams calculated within ChPT, through the Lippmann–Schwinger equation (see also [19, 29] for earlier references). In addition, we use the experimental values of the scattering lengths given in [30]. As we see from the table, the corrections at  $O(\sqrt{\delta})$  are indeed huge – they amount up to a few tens of percent. More precise predictions are not possible, because the scattering lengths themselves are not very well known. On the other hand, instead of  $\frac{1}{2}(a_0 + a_1)$  as contained in the original Deser formula, one might determine the combination of these scattering lengths  $\mathcal{T}_{KN}^{(0)}$ . This combination, which already includes the big corrections at  $O(\sqrt{\delta})$ , can be extracted from data with a much better accuracy. As one may observe from Table 1, the convergence of the expansion in the parameter  $\sqrt{\delta}$  is rather good.

Note also that if one uses the scattering lengths  $a_0 = -2.24 + 1.94i$ ,  $a_1 = 0.54 + 0.54i$  (in fm) as given in [19], then the convergence of the series in  $\sqrt{\delta}$  is significantly worse. More precisely, one gets  $-0.85 + 1.24i$ ,  $-1.28 + 1.82i$ ,  $-1.17 + 2.12i$ ,  $-1.04 + 2.16i$ ,  $\dots - 1.00 + 2.09i$  (in the notation employed in Table 1). It can be argued that this result contradicts the general expectation for the size of the isospin-breaking corrections: one sees that in these series the corrections at  $O(\delta^{3/2})$  amount approximately to 15–20% in the real part of the amplitude. Moreover, this result also contradicts our derivation of the energy shift: there is no justification for neglecting the  $O(\delta^{3/2})$  terms in the bound-state energy, if in the amplitude their contribu-

<sup>7</sup> The scattering lengths  $a_0, a_1$  are not displayed separately in [20]. We thank J. Oller for providing these values.

tions are so large. However, we would like to stress that in the approach used in [19], the straightforward introduction of a cutoff to regularize the unitarity resummation violates chiral symmetry, since the amplitudes are not matched to the chiral expansion<sup>8</sup>. The considerably larger values of the scattering lengths than those determined by the experiment [30], which in its turn cause problems concerning the convergence of the series, might have been resulted from this implicit violation of chiral symmetry. Such a problem does not arise in the calculation of [20] since an explicit matching to the ChPT amplitudes is performed and the regularization is done employing subtracted dispersion relations. That this procedure leads to reasonable scattering parameters was demonstrated explicitly for the case of pion–nucleon scattering in [32].

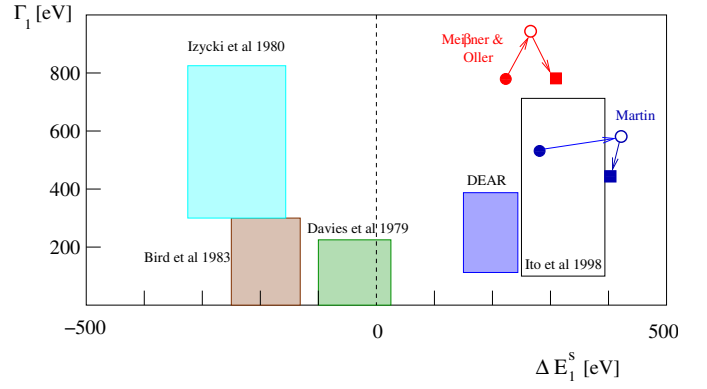
We further investigate the magnitude of  $O(\delta)$  corrections in (19). The Coulomb corrections that are amplified by  $\ln \alpha$  are sizable but smaller than those due to the unitarity cusp: for the choice of scattering lengths from [20, 30], the real part of the correction term in the ground state is [9%, 15%], respectively. Again, we do not need to know this number very accurately; since the Coulomb correction depends on the scattering lengths  $a_0, a_1$  only, we can use the modified Deser relation which includes the Coulomb term to determine  $\mathcal{T}_{KN}^{(0)}$ .

Below, we briefly consider other corrections contained in (19). The calculation of the quantity  $\delta\mathcal{T}_{KN}$  proceeds analogously to the  $\pi N$  case. This quantity starts at order  $p^2$  in ChPT. Further, according to (17), it is equal to the isospin-breaking part of  $\mathcal{T}_{KN}$  at this order. In the actual calculations we have used strong (see, e.g. [33–36]) and electromagnetic meson–baryon Lagrangians of  $SU(3) \times SU(3)$  ChPT (for the construction principles, see [37]):

$$\begin{aligned} \mathcal{L}_2 = & b_0 \text{Tr}(\bar{B}B) \text{Tr}(\chi_+) + b_D \text{Tr}(\bar{B}\{\chi_+, B\}) \\ & + b_F \text{Tr}(\bar{B}\{\chi_+, B\}) + F_0^2 G_1 \text{Tr}(Q_+^2) \text{Tr}(\bar{B}B) \\ & + F_0^2 G_2 \text{Tr}(Q_-^2) \text{Tr}(\bar{B}B) + F_0^2 G_3 \text{Tr}(\bar{B}Q_+^2 B) \\ & + F_0^2 G_4 \text{Tr}(\bar{B}Q_+ B Q_+) + F_0^2 G_5 \text{Tr}(\bar{B}B Q_+^2) \\ & + F_0^2 G_6 \text{Tr}(\bar{B}Q_-^2 B) + F_0^2 G_7 \text{Tr}(\bar{B}Q_- B Q_-) \\ & + F_0^2 G_8 \text{Tr}(\bar{B}B Q_-^2) + \text{terms with derivatives,} \end{aligned} \quad (20)$$

where  $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$  and  $Q = e \text{diag}(2/3, -1/3, -1/3)$  are the quark mass matrix and the charge matrix, respectively,  $B$  stands for the baryon octet field,  $\chi_+ = 2B_0(u\mathcal{M}u + u^\dagger\mathcal{M}u^\dagger)$ ,  $Q_\pm = \frac{1}{2}(uQu^\dagger \pm u^\dagger Qu)$ , and  $U = u^2 = \exp\left(\frac{i}{F_0}\Phi\right)$ , with  $\Phi$  being the pseudoscalar boson octet field. Here,  $F_0$  is the Goldstone boson decay constant (in the chiral limit) and  $B_0 = |\langle 0|\bar{q}q|0\rangle|/F_0^2$  measures the strength of the quark–antiquark condensate (in the chiral limit). Further,  $b_0, b_D, b_f$  and  $G_i$  denote  $O(p^2)$  strong and electromagnetic low-energy constants (LECs), respectively. Note that since our definition of the isospin limit involves the physical masses of the charged particles, the terms

<sup>8</sup> For a recent discussion on the use of cutoff regularization in chiral effective field theories, see [31].



**Fig. 3.** Predictions of the ground-state strong shift  $\Delta E_1^s$  and width  $\Gamma_1$ . Filled circles correspond to using the Deser formula (2), empty circles to using  $\mathcal{T}_{KN}^{(0)}$  instead of  $\frac{1}{2}(a_0 + a_1)$  in this formula, and filled boxes to our final formula (19) with  $\delta\mathcal{T}_{KN} = \delta_n^{\text{vac}} = 0$

with derivatives of the kaon fields do not contribute to the isospin-breaking part of the  $K^-p$  elastic amplitude at  $O(p^2)$ . This quantity is equal to

$$\begin{aligned} \delta\mathcal{T}_{KN} = & \frac{4(\Delta M_K^2)_{\text{em}}}{F_0^2} (b_0 + b_D) \\ & + e^2 \left\{ -G_1 + G_2 - \frac{2}{3} G_3 + \frac{1}{2} G_4 \right. \\ & \left. - \frac{1}{3} G_5 + \frac{1}{2} G_6 + \frac{1}{2} G_8 \right\}, \quad (21) \end{aligned}$$

where the subscript “em” denotes the electromagnetic mass shift. Using Dashen’s theorem, we can replace  $(\Delta M_K^2)_{\text{em}}$  by  $(\Delta M_\pi^2)_{\text{em}} = \Delta M_\pi^2 + \dots$  at this order. Further, in the numerical estimates we use  $b_0 = -0.517 \text{ GeV}^{-1}$ ,  $b_D = 0.066 \text{ GeV}^{-1}$  from [35]. For the unknown electromagnetic LECs the order-of-magnitude estimate  $e^2 F_0^2 |G_i| \leq \alpha m_p / 2\pi$  was used. With these numerical values one gets  $\delta\mathcal{T}_{KN}/\mathcal{T}_{KN}^{1.o.} = (-0.5 \pm 0.4)\%$  at  $O(p^2)$ , where  $\mathcal{T}_{KN}^{1.o.} = M_{K^+}/F_0^2$  is the isospin-symmetric part of the  $K^-p$  amplitude at the leading order. The fact that the uncertainty in the isospin-breaking part of the  $KN$  amplitude turns out to be smaller than in the  $\pi N$  amplitude [17] is not surprising: the quantities  $\delta\mathcal{T}_{KN}$  and  $\delta\mathcal{T}_{\pi N}$  are of the same order of magnitude, whereas the isospin-symmetric part in the  $KN$  amplitude is increased by a factor  $M_{K^+}/M_{\pi^+}$  and additionally by a group-theoretical factor of 2. The results are anyway to be taken with a grain of salt: it needs to be seen how the results are changed in higher orders in the chiral expansion. Finally, we note that the value  $\delta_1^{\text{vac}} = 0.87\%$  given in [27] suggests that at present accuracy one may well include this term in the systematic error.

In order to visualize the size of the isospin-breaking effects, in Fig. 3 we plot the theoretical predictions corresponding to the scattering lengths from [20, 30] versus the old and new experimental measurements of the energy spectrum of kaonic hydrogen [2–6]. As we immediately observe from the plot, the use of the lowest-order Deser

formula (2) cannot be justified anymore: both cusp effect and Coulomb corrections have a size comparable with the present precision of the DEAR experiment and should be taken into account on extracting S-wave  $KN$  scattering lengths from the experimental data. Note an apparent discrepancy between the recent DEAR measurements and the predictions obtained by using the scattering lengths from [20, 30]. Moreover, one sees that when the isospin-breaking corrections are applied, the results move away from the DEAR measurements. We conclude that further investigations are needed in order to shed light on this interesting issue.

## 5 Summary and outlook

In this paper we derived the formal expression for the strong shifts of the energy levels in kaonic hydrogen in QCD, up-to-and-including  $O(\delta^4)$  in the isospin-breaking parameter  $\delta \sim \alpha, m_d - m_u$ . The use of the non-relativistic effective Lagrangian approach allows one to treat that otherwise extremely complicated problem with a surprising ease. We discover that large isospin-breaking corrections arise, in particular, due to the following sources:

- (a)  $s$ -channel rescattering with the  $\bar{K}^0 n$  intermediate state (cusp effect), and
- (b) Coulomb corrections that are non-analytic in  $\alpha$ . We further prove that the remaining corrections are analytic in  $\delta$  at  $O(\delta)$ . Examining some of these corrections, on the other hand, we do not find a big effect – the obtained values are at the percent level, which one expects to be a typical size of isospin breaking in QCD.

The present status of corrections in kaonic hydrogen can be summarized by (19). Instead of the combination  $\frac{1}{2}(a_0 + a_1)$  which enters in the original Deser formula (2), we propose to focus on the extraction of the quantity  $\mathcal{T}_{KN}^{(0)}$  from the experimental data. The reason for this is that  $\mathcal{T}_{KN}^{(0)}$  already includes the dominant non-analytic corrections in a parameter-free form. The remaining analytic corrections at  $O(\delta)$  are contained in the quantities  $\delta\mathcal{T}_{KN}$  and  $\delta_n^{\text{vac}}$ . The evaluation of  $\delta\mathcal{T}_{KN}$  within ChPT could be interesting, but possibly complicated due to the expansion in the strange quark mass. At the present stage, in the absence of such calculations, the best is to include  $\delta\mathcal{T}_{KN}$  in the estimate of the systematic error. From the above discussion one may hope that the effect from  $\delta\mathcal{T}_{KN}$  should not exceed a few percent, which is a natural size of electromagnetic corrections.

*Acknowledgements.* The authors thank J. Gasser, A. Ivanov, C. Petrascu, J. Schweizer and P. Zemp for useful discussions. We are particularly grateful to Jürg Gasser for a careful reading of the manuscript.

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